ARITHMETIC LOCAL SYSTEMS AND p-ADIC HODGE THEORY

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Cohomology groups $H^n(X(\mathbb{C}), \mathbb{Z})$ of a complex algebraic variety X admit a remarkably rich array of additional structures: (mixed) Hodge structure, Galois action coming from étale cohomology, crystalline Frobenii, and the list goes on. The presence of these structures leads to interesting constraints on the topology of algebraic varieties and maps between them.

Another[1] type of structure that has proven to be very useful in algebraic geometry is *monodromy*. If a smooth proper variety X is established as a fiber $X = \mathcal{X}_s$ of a smooth proper morphism $\pi : \mathcal{X} \to S$ of complex algebraic varieties over a point $s \in S(\mathbb{C})$, then the groups $H^n(X(\mathbb{C}), \mathbb{Z})$ have a natural action of the fundamental group $\pi_1(S(\mathbb{C}), s)$ of the base of the family.

In this expository survey, we focus on the action of the Galois group on cohomology coming from the theory of étale cohomology, and on its interaction with monodromy. This interaction has a long history of fruitful applications. For example, Deligne's proof of Weil's conjectures, which is a statement about the Galois action on the cohomology of a variety, proceeds by establishing the variety at hand as the total space of a family with non-trivial enough monodromy representation, and studying the variation of the Galois action on the cohomology of the fibers of the family. One general point that we will try to make is that many seemingly unrelated structures on cohomology can be recovered from the Galois action.

In Section 1 we review the definition of étale fundamental group and the Galois action on it, and give two applications of this structure to the topology of maps between complex algebraic varieties. In Section 2 we state a conjectural arithmetic characterization of monodromy representations coming from families of algebraic varieties. In Section 3 we briefly introduce some of the constructions of p-adic Hodge theory, and discuss an application to the construction of a mixed Hodge structure on the cohomology of open singular complex varieties. Finally, in Sections 4 and 5 we discuss evidence for Conjecture 1 coming from Langlands correspondence and p-adic Hodge theory, respectively.

None of the material presented here is original, and we have attempted to give proper references, though our attributions are at best approximate, especially for the more foundational results. We are only able to scratch the surface, and refer the reader to excellent recent surveys such as [Esn23], [Lit24] for a more comprehensive discussion of some other aspects of this story.

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[1] In fact, we will see that monodromy was already essentially mentioned in the list of structures in the previous paragraph

1. Étale cohomology and étale fundamental group

A fundamental result of Grothendieck and his collaborators is that singular cohomology groups $H^n(X(\mathbb{C}),\mathbb{Z})$ of a complex algebraic variety X can be reconstructed by a purely algebraic procedure from X via the theory of étale site and étale cohomology, if one is content with replacing the coefficient group $\mathbb{Z}$ with a finite group $\mathbb{Z}/n$ or the group $\mathbb{Z}_\ell$ of ℓ-adic integers for some prime number ℓ.

This theory has a much wider scope than algebraic varieties over complex numbers and provides, for any scheme X, the $\mathbb{Z}_\ell$-modules

$$H^n_{\mathrm{et}}(X,\mathbb{Z}_\ell) \tag{1.1}$$

indexed by integers $n \geq 0$, for every prime ℓ, satisfying the following properties:

(1) If X is a separated scheme of finite type over $\mathbb{C}$, then there is a natural isomorphism $H^n_{\mathrm{et}}(X,\mathbb{Z}_\ell) \simeq H^n(X(\mathbb{C}),\mathbb{Z}_\ell)$ with singular cohomology groups of the space $X(\mathbb{C})$ with its complex-analytic topology.
(2) If $k \subset k'$ is an extension of algebraically closed fields with $\ell \neq \mathrm{char}(k)$ then for a finite type scheme X over k the natural map $H^n_{\mathrm{et}}(X,\mathbb{Z}_\ell) \to H^n_{\mathrm{et}}(X_k \times k',\mathbb{Z}_\ell)$ is an isomorphism.

Note that this theory is absolute, that is groups $H^n_{\mathrm{et}}(X,\mathbb{Z}_\ell)$ naturally depend just on the scheme X, without reference to the structure map to any base. In particular, if X_0 is a scheme over a field F then étale cohomology $H^n_{\mathrm{et}}(X_0\times_F \overline{F},\mathbb{Z}_\ell)$ of its base change to an algebraic closure $\overline{F}$ of F has a natural action of the absolute Galois group $G_F := \mathrm{Aut}(\overline{F}/F)$ of F.

Any finite type scheme X over $\mathbb{C}$ admits a descent X_0 to a finitely generated subfield $F \subset \mathbb{C}$. Combining properties (1) and (2) above we have an isomorphism

$$H^n(X(\mathbb{C}),\mathbb{Z}_\ell) \simeq H^n_{\mathrm{et}}(X,\mathbb{Z}_\ell) \simeq H^n_{\mathrm{et}}(X_{0,\overline{F}},\mathbb{Z}_\ell) \tag{1.2}$$

with the last group having a natural action of the Galois group G_F, that induces an action on the singular cohomology, by transport of structure. The choice of a descent X_0 is not unique, but any two such descents X_0, X_0' over subfields $F, F' \subset \mathbb{C}$ become isomorphic over a common finite extension $F'' \supset F, F'$, and in particular the resulting actions of G_F and $G_{F'}$ are identified on a common finite index subgroup.

It turns out to be fruitful to think of this Galois action as an additional structure intrinsic to a complex algebraic variety – the mild dependence on the descent will be erased by some of the constructions one makes.

Étale fundamental group. We will not review the definition of étale cohomology here, and will limit ourselves to recalling the definition of étale fundamental group, which in particular captures the value of étale cohomology in degree 1. Many foundational results about étale cohomology are reduced to the case of the étale fundamental group that can be accessed by geometric arguments.

Given a connected scheme X equipped with a geometric point x, that is with a map from the spectrum of an algebraically closed field to X, consider the following category of covers of X. Its objects are pairs (Y, y) where Y is a scheme equipped with a finite étale map $f : Y \to X$ and y is a geometric point of Y such that $f(y)$ is equivalent to x.

Recall that a finite étale map $f : Y \to X$ is a Galois covering if the group of automorphisms $\mathrm{Aut}(Y/X)$ of Y relative to X has order equal to the degree of f. If a Galois covering $f : Y \to X$ factors as $Y \to Y' \xrightarrow{f'} X$ where $f' : Y' \to X$ is another Galois covering, every automorphism of Y over X is compatible with a unique automorphism of Y', hence defining a group homomorphism $\mathrm{Aut}(Y/X) \to \mathrm{Aut}(Y'/X)$. We now define the étale fundamental group as the inverse limit of these automorphism groups over the category of pointed Galois covers of (X, x):

$$\pi_1^{\mathrm{et}}(X, x) := \lim_{(Y,y)} \mathrm{Aut}(Y/X) \tag{1.3}$$

Note that the chosen points y of schemes Y do not play a role in the definition of the functor we are taking the limit of, but it is crucial to take the limit over the category of pointed covers: the limit over the category of covers without a base point would give the abelinization of the correct fundamental group.

This construction captures degree 1 étale cohomology of an arbitrary scheme, and partially recovers the topological fundamental group:

(1) For a connected finite type scheme X over $\mathbb{C}$ with a geometric point above a $\mathbb{C}$-point $x \in X(\mathbb{C})$ the group $\pi_1^{\mathrm{et}}(X, x)$ is naturally isomorphic to the profinite completion $\pi_1(\widehat{X(\mathbb{C}), x})$ of the topological fundamental group.
(2) For each prime ℓ, for any connected scheme X the 1st cohomology group $H^1_{\mathrm{et}}(X, \mathbb{Z}_\ell)$ is isomorphic to the group of continuous homomorphisms $\mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{et}}(X, x), \mathbb{Z}_\ell)$.

For a scheme X over a field F equipped with a geometric base point above an F-point $x \in X(F)$ we get a natural action of the Galois group G_F on the profinite fundamental group $\pi_1^{\mathrm{et}}(X_{\overline{F}}, x)$, analogously to the Galois action on étale cohomology. Let us compute this action in two examples:

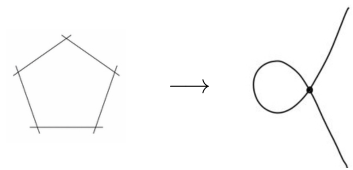
Example 1.1. Let F be any field of characteristic zero, and consider the punctured affine line $X = \mathbb{G}_{m,F}$. Using Riemann-Hurwitz formula one can check that every connected finite étale cover of $X_{\overline{F}} = \mathbb{G}_{m,\overline{F}}$ is of the form $X_n = \mathbb{G}_{m,\overline{F}} \xrightarrow{t \mapsto t^n} \mathbb{G}_{m,\overline{F}}$ for some integer $n \geq 1$. Every such cover is Galois, with the Galois group naturally isomorphic to the group of n-th roots of unity $\mu_n(\overline{F})$ in $\overline{F}$, with $\zeta \in \mu_n(\overline{F})$ acting via $t \mapsto \zeta \cdot t$. Hence the étale fundamental group of $\mathbb{G}_{m,\overline{F}}$ with respect to any base point $a \in \mathbb{G}_m(F)$ is

$$\pi_1^{\mathrm{et}}(\mathbb{G}_{m,\overline{F}}, a) = \lim_n \mu_n(\overline{F}) \simeq \widehat{\mathbb{Z}}(1) \tag{1.4}$$

Each $\mu_n(\overline{F})$ is of course isomorphic to $\mathbb{Z}/n\mathbb{Z}$ as an abelian group, but the natural action of the Galois group $\mathrm{Gal}(\overline{F}/F)$ on $\pi_1^{\mathrm{et}}(\mathbb{G}_{m,\overline{F}}, a)$ is via its action on the roots of unity on $\overline{F}$. In particular, this action is non-trivial unless $\mu_\infty(\overline{F})$ is entirely contained in F.

Example 1.2. Let Y be a proper rational curve with a single (split) nodal singularity, over a field $F \subset \mathbb{C}$ of characteristic 0. Explicitly, Y is isomorphic to the plane curve cut out by equation $zy^2 = x^2(x + z)$ in $\mathbb{P}^2_{[x:y:z]}$ and can be presented as the categorical quotient $\mathbb{P}^1_F/(0 \sim \infty)$. Note that $Y(\mathbb{C})$ is homotopy equivalent to $S^1 \vee S^2$, so by the comparison isomorphism with the profinite completion of the topological fundamental group, we expect $\pi_1^{\mathrm{et}}(Y_{\overline{F}})$ to be isomorphic to $\widehat{\mathbb{Z}}$, just like in the case of $\mathbb{G}_{m,\overline{F}}$.

Indeed, $Y_{\overline{F}}$ has just one isomorphism class of connected etale covers of degree n, for each $n \geq 1$. For $n > 1$ the unique such cover is a circle of n projective lines:



The automorphism group of this cover is isomorphic to $\mathbb{Z}/n$, with all automorphisms given by rotations of the circle. In particular, all of these automorphisms are already defined over the base field F, so the natural action of the Galois group $\mathrm{Gal}(\overline{F}/F)$ on

$$\pi_1^{\mathrm{et}}(Y_{\overline{F}}) = \lim_n \mathbb{Z}/n = \widehat{\mathbb{Z}} \tag{1.5}$$

is trivial.

Corollary 1.3. *For a smooth algebraic variety X over $\mathbb{C}$ any map $f : Y \to X$ from a proper rational curve with a single node induces the zero map $f_* : \mathbb{Q} \simeq H_1(Y(\mathbb{C}), \mathbb{Q}) \to H_1(X(\mathbb{C}), \mathbb{Q})$ on rational homology in degree 1.*

Proof. We can choose a map $f_0 : Y_0 \to X_0$ of varieties over a finitely generated subfield $F \subset \mathbb{C}$ descending f, where Y_0 is again a rational curve with a single split node, and X_0 is necessarily smooth. It suffices to check that the induced map $f_0^* : H^1_{\mathrm{et}}(X_{0,\overline{F}}, \mathbb{Q}_\ell) \to H^1(Y_{0,\overline{F}}, \mathbb{Q}_\ell)$ on 1st étale cohomology is zero. Crucially, this map commutes with the action of G_F, but this action on cohomology of $Y_{0,\overline{F}}$ is trivial by Example 1.2 while the action on the cohomology of $X_{0,\overline{F}}$ in positive degrees has no non-zero invariants by Theorem 1.4 below. □

The key property of the Galois action on étale cohomology used above is the final Weil conjecture, proved by Deligne:

Theorem 1.4. *For a smooth variety X over a finitely generated field F of $\mathrm{char}(F) \neq \ell$ there exists a finitely generated subalgebra $R \subset F$ with $F = \mathrm{Frac}\, R$ such that the action of G_F on $H^n_{et}(X_{\overline{F}}, \mathbb{Q}_\ell)$ factors through an action of $\pi_1^{et}(\mathrm{Spec}\, R)$. After replacing R by a localization, the Frobenius element $\mathrm{Fr}_x \in \pi_1^{et}(\mathrm{Spec}\, R)$ at every closed point $x \in |\mathrm{Spec}\, R|$ acts in $H^n_{et}(X_{\overline{F}}, \mathbb{Q}_\ell)$ via an endomorphism with eigenvalues that are Weil numbers of weight $\geq n$.*

If X is also proper, the eigenvalues (for a small enough choice of $\mathrm{Spec}\, R$) are Weil numbers of weight equal to n.

Quasi-unipotence of local monodromy. We will now start discussing how the Galois action on cohomology interacts with the monodromy action, and, as an illustration, will review Grothendieck's proof of the local monodromy theorem. Recall that for a smooth proper map $f : X \to S$ of complex algebraic varieties there is a natural action of the fundamental group of the base $\pi_1(S(\mathbb{C}), s)$ on the cohomology of the fiber $H^k(X_s(\mathbb{C}), \mathbb{Z})$ at a given point $s \in S(\mathbb{C})$.

While this action manifestly depends only on the homotopy class of the map $X(\mathbb{C}) \to S(\mathbb{C})$, it often contains non-trivial information about the algebraic structure of the family. Here is a simple example of this phenomenon:

Proposition 1.5. *Let $f : \mathcal{E} \to S$ be a family of elliptic curves over a connected variety S over $\mathbb{C}$. If the monodromy representation $\rho_{\mathcal{E}} : \pi_1(S(\mathbb{C}), s) \to GL(H^1(\mathcal{E}_s, \mathbb{Q})) \simeq GL_2(\mathbb{Q})$*

has infinite image, then the family $\mathcal{E}$ is completely determined by the conjugacy class of $\rho_{\mathcal{E}}$.

Proof. This is a special case of [Del71, Proposition 4.4.12] whose assumptions are satisfied, because $\rho_{\mathcal{E}}$ is absolutely irreducible under our condition. Indeed, $\rho_{\mathcal{E}}$ is always semi-simple by [Del71, 4.2.6], and if $\rho_{\mathcal{E}}$ is not absolutely irreducible, $\rho_{\mathcal{E}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is a direct sum of characters, but every character of geometric origin has finite order as we will see below in Proposition 2.6. □

Suppose now that S is a non-empty open in a smooth proper curve $\overline{S}$ over $\mathbb{C}$. We will look at the monodromy action of a specific class of elements of $\pi_1(S(\mathbb{C}), s)$. For one of the finitely many points $x \in (\overline{S} \setminus S)(\mathbb{C})$ let $\gamma_x \in \pi_1(S(\mathbb{C}), s)$ be an element obtained by traveling from s to a point in a punctured neighborhood of x along any path, circling around x once in the clockwise direction with respect to the orientation defined by the complex structure, and then traveling back to s along the very same path. Element γ_x in general depends on the choice of the path, but its conjugacy class is completely well-defined.

The action of γ_x in monodromy representations on the cohomology of a family of varieties turns out to be very special:

Theorem 1.6. *Let S be a non-empty open in a smooth proper curve $\overline{S}$ over $\mathbb{C}$. For any smooth proper morphism $f : X \to S$, and any boundary point $x \in \overline{S} \setminus S$ the action of γ_x on $H^k(X_s(\mathbb{C}), \mathbb{Q})$ is quasi-unipotent, that is the eigenvalues of $\rho_X(\gamma_x)$ are roots of unity.*

Proof. Our first maneuver, already familiar from the proof of Corollary 1.3, is to introduce arithmetic to the problem, by descending the family $X \to S$ to a smooth proper family $X_0 \to S_0$ over a finitely generated subfield $F \subset \mathbb{C}$. We take F to be large enough to be able to assume that x and s descend to F-points of $\overline{S}_0$.

The cohomology group $H^k(X_s(\mathbb{C}), \mathbb{Q}_\ell)$ of the fiber is now equipped with two structures: the action of the topological fundamental group $\pi_1(S(\mathbb{C}), s)$ and the action of the Galois group G_F via the isomorphism $H^k(X_s(\mathbb{C}), \mathbb{Q}_\ell) \simeq H^k_{\text{et}}(X_{0,s} \times_F \overline{F}, \mathbb{Q}_\ell)$.

These actions do not commute with each other, but rather interact as follows. There is an action of the étale fundamental group $\pi_1^{\text{et}}(S_0, s)$ of the scheme S_0 (as a scheme over F, without base changing to $\overline{F}$) on $H^k_{\text{et}}(X_{0,s} \times_F \overline{F}, \mathbb{Q}_\ell)$ that encompasses both the action of G_F and $\pi_1(S(\mathbb{C}), s)$. Étale fundamental group $\pi_1^{\text{et}}(S_0, s)$ with respect to a geometric base point lying above a rational F-point is identified with the semi-direct product:

$$\pi_1^{\text{et}}(S_0, s) \simeq G_F \ltimes \pi_1^{\text{et}}(S_{0,\overline{F}}, s).$$

This is a reformulation of the 'homotopy exact sequence' [Gro63, Corollaire X.2.2] and stems from the fact that every finite étale cover of $S_0 \times_F \overline{F}$ descends to a finite étale cover of $S_0 \times_F F'$ for some finite extension F' of F.

In turn, $\pi_1^{\text{et}}(S_{0,\overline{F}}, s)$ is the pro-finite completion of the topological fundamental group $\pi_1(S(\mathbb{C}), s)$. The fact that the monodromy representation of $\pi_1(S(\mathbb{C}), s)$ into $GL(H^k(X_s(\mathbb{C}), \mathbb{Q}_\ell))$ extends to its pro-finite completion is in fact automatic, because this action preserves a $\mathbb{Z}_\ell$-lattice. But the crucial new piece of structure is that the monodromy representation further extends to the above semi-direct product.

The action of G_F on $\pi_1\widehat{(S(\mathbb{C}), s)}$ is in general very complicated, and getting a tangible description of it would solve several central problems of arithmetic geometry. But elements of γ_x that we are interested in are special in that we can compute the action of the Galois group on them, and crucially it is non-trivial:

Lemma 1.7 ([SGA7II, Exposé XIV, 1.1.10])**.** *For every element of the Galois group $g \in G_F$ we have that $g(\gamma_x) \in \pi_1(\widehat{S(\mathbb{C})}, s)$ is conjugate to $\gamma_x^{\chi_{\mathrm{cycl}}(g)}$. Here $\chi_{\mathrm{cycl}}(g) \in \widehat{\mathbb{Z}}^\times$ is the value of the cyclotomic character on g, and what we mean by $\gamma_x^{\chi_{\mathrm{cycl}}(g)}$ is the image of $\chi_{\mathrm{cycl}}(g) \in \widehat{\mathbb{Z}}$ under the pro-finite completion of the map $\mathbb{Z} \xrightarrow{1 \mapsto \gamma_x} \pi_1(S(\mathbb{C}), s)$.*

Proof idea of Lemma 1.7. Computing the conjugacy class of $g(\gamma_x)$ is a problem that can be solved locally in a formal neighborhood of the point x, so one reduces to computing the Galois action on the étale fundamental group of $\operatorname{Spec} \overline{F}((t))$, which in turn is equivalent to the computation from Example 1.1. □

We can now conclude that $\rho_X(\gamma_x)$ is quasi-unipotent. As the field F is finitely generated over $\mathbb{Q}$, it contains only finitely many roots of unity, that is the image of the cyclotomic character $G_F \to \widehat{\mathbb{Z}}^\times$ is a finite index open subgroup in the target. In particular, we may choose an element $g \in G_F$ such that the composition $G_F \xrightarrow{\chi_{\mathrm{cycl}}} \widehat{\mathbb{Z}}^\times \to \mathbb{Z}_\ell^\times$ sends g to an element $r \in \mathbb{Z}_\ell^\times$ for some integer $\mathbb{Z} \ni r \neq \pm 1$.

Eigenvalues of $\rho_X(\gamma_x)$ on $H^k(X_s(\mathbb{C}), \mathbb{Q}_\ell)$ are a priori invertible elements of $\overline{\mathbb{Z}}_\ell \subset \overline{\mathbb{Q}}_\ell$. The fact that ρ_X extends to $G_F \ltimes \pi_1(\widehat{S(\mathbb{C})}, s)$ implies that $\rho_X(\gamma_x)$ is conjugate to $\rho_X(g(\gamma_x))$ which by Lemma 1.7 is conjugate to $\rho_X(\gamma_x)^{\chi_{\mathrm{cycl}}(g)}$. By our choice of the element g, this implies that the set of eigenvalues of $\rho_X(\gamma_x)$ is invariant under raising to the rth power. This forces all of them to be roots of unity, as desired. □

It will be convenient for us to have another term for representations of étale fundamental group:

Definition 1.8. For a connected scheme X with a geometric base point x an *étale local system* on X with coefficients in any of the topological rings $R = \mathbb{Z}_\ell, \mathbb{Z}/\ell^d, \mathbb{Q}_\ell, \overline{\mathbb{Q}}_\ell$ is a conjugacy class of continuous representations $\pi_1^{\mathrm{et}}(X, x) \to GL_n(R)$.

This is not formally obvious from the way we set up the definition, but the category of étale local systems on X is functorially independent of the choice of the base point. In particular, a local system on X can be restricted to every field-valued point $\operatorname{Spec} L \to X$ to obtain a representation of the Galois group G_L.

2. Classification of local systems of geometric origin

Note that the only property of the monodromy representation of $\pi_1(S(\mathbb{C}), s)$ on $H^k(X_s(\mathbb{C}), \mathbb{Q})$ that was used in the proof of Theorem 1.6 is that it extends to a representation of $\pi_1^{\mathrm{et}}(S_0, s)$ with coefficients in $\mathbb{Q}_\ell$. Let us give this property a name:

Definition 2.1 ([Lit18, Definition 1.1.1])**.** For a connected finite type scheme S over $\mathbb{C}$ a representation $\rho : \pi_1(S(\mathbb{C}), s) \to GL_n(\overline{\mathbb{Q}}_\ell)$ is *arithmetic* if there exists a finitely generated subfield $F \subset \mathbb{C}$ and a descent S_0 of S to F such that ρ extends to a continuous representation $\widetilde{\rho} : \pi_1^{\mathrm{et}}(S_0, s) \to GL_n(\overline{\mathbb{Q}}_\ell)$.

The fact that ρ extends to the profinite completion $\pi_1^{\mathrm{et}}(S_{\mathbb{C}}, s) \simeq \pi_1(\widehat{S(\mathbb{C})}, s)$ of the topological fundamental group is equivalent to ρ being conjugate to a representation landing in $GL_n(\mathcal{O}_E)$, for a finite extension E of $\mathbb{Q}_\ell$. Extendability to the fundamental group of S_0 is the most serious part of the 'arithmetic' property.

Remark 2.2. One might reasonably object that it is strange to treat arithmeticity as a property, rather than keeping track of the chosen extension of ρ. For now, let us just point out that this ambiguity is not very serious if ρ is irreducible: if $\widetilde{\rho}, \widetilde{\rho}'$ are two extensions, for a given field F, then Schur's lemma implies that $\widetilde{\rho}'$ is conjugate to $\widetilde{\rho} \otimes \chi$ for some character $\chi : G_F \to \overline{\mathbb{Q}}_\ell^\times$ of the Galois group. In particular, an irreducible arithmetic representation has a unique extension to $\pi_1^{\mathrm{et}}(S_0)$ as a projective representation.

This is of course an interesting property only if the Galois action on the fundamental group of $S_{0,\overline{F}}$ is sufficiently non-trivial. We already saw that this is the case for $S_0 = \mathbb{G}_{m,F}$ but let us point out a much stronger non-triviality property of this action:

Theorem 2.3 ([Bel79, Corollary on p. 256], [Gro])**.** *For any choice of base point $a \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ every non-trivial element $g \in G_{\mathbb{Q}}$ acts on $\pi_1^{et}(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}, a)$ via a non-inner automorphism.*

Let us now discuss more systematically the relation between arithmetic representations and monodromy representation on cohomology of families of varieties.

Definition 2.4. For a normal connected algebraic variety S over $\mathbb{C}$ a representation $\rho : \pi_1(S(\mathbb{C}), s) \to GL_n(\overline{\mathbb{Q}}_\ell)$ is of *geometric origin* if there exists a non-empty Zariski open $U \subset S$ such that the restriction $\pi_1(U(\mathbb{C})) \twoheadrightarrow \pi_1(S(\mathbb{C})) \to GL_n(\overline{\mathbb{Q}}_\ell)$ is a subquotient of the monodromy representation on the k-th cohomology of the fiber of a smooth proper family $f : X \to U$, for some k.

Remark 2.5.

(1) We restrict to the case of a normal S to ensure that passing to an open U induces a surjection on fundamental groups [FL81, (0.7)B]. In particular, the isomorphism class of ρ is completely determined by $\rho|_{\pi_1(U(\mathbb{C}))}$.

(2) It is not currently known whether every representation of geometric origin is a subquotient of the cohomology of a smooth proper family over all of S, but that seems rather unlikely. One reason to allow the flexibility of restricting to an open is that Drinfeld-L. Lafforgue's Theorem 4.1 proves that a given local system is of geometric origin in that sense.

(3) The definition relies on the choice of a prime ℓ – we work with $\overline{\mathbb{Q}}_\ell$ coefficients for the convenience of relating this notion to arithmetic representations later on. But the notion of geometric origin is independent of ℓ in the following sense: if ρ is of geometric origin, then it is conjugate to a representation factoring through $GL_n(\overline{\mathbb{Q}}) \subset GL_n(\overline{\mathbb{Q}}_\ell)$. Conversely, if a representation $\rho : \pi_1(S(\mathbb{C}), s) \to GL_n(\overline{\mathbb{Q}})$ is of geometric origin when viewed as a $\overline{\mathbb{Q}}_\ell$-valued representation for one ℓ, the same is true for all values of ℓ.

(4) In this definition 'subquotient' can be replaced by 'direct summand', as the representation of $\pi_1(U(\mathbb{C}))$ on $H^k(X_s, \mathbb{Q})$ for a smooth proper family $X \to U$ is always semi-simple. This can be proven either using that this representation underlies a polarized variation of Hodge structures [Del87, 1.12], or using that it gives rise to a *pure* local system on the reduction of U over some finite field [Del80, Corollaire 3.4.13].

As we discussed in the proof of Theorem 1.6, every representation of the form $H^k(X_s, \overline{\mathbb{Q}}_\ell)$ is arithmetic, and the flexibility of choosing the field F implies that moreover every representation of geometric origin is arithmetic [Lit21, Proposition 3.1.7]. It turns out to be reasonable to conjecture the converse:

Conjecture 1 ([Pet23, Conjecture 1bis]). *For a normal finite type scheme S over $\mathbb{C}$ a semi-simple representation of $\pi_1(S(\mathbb{C}),s)$ is arithmetic if and only if it is of geometric origin.*

One of the goals of this survey is to discuss evidence for this conjecture which comes from two disparate directions in arithmetic geometry. Let us begin by proving the conjecture for rank 1 representations:

Proposition 2.6 ([Del80, Théorème 1.3.1]). *For a normal scheme S, a character $\rho : \pi_1(S(\mathbb{C}),s)\to\overline{\mathbb{Q}}_\ell^\times$ is arithmetic if and only if it has finite image.*

Proof. Suppose that ρ extends to a character of $G_F\ltimes\pi_1(\widehat{S(\mathbb{C})},s)\xrightarrow{\widetilde{\rho}}\mathcal{O}_E^\times\subset\overline{\mathbb{Q}}_\ell^\times$ of the arithmetic fundamental group, where E is some finite extension of $\mathbb{Q}_\ell$, and the action of G_F is defined by some descent of S over a finitely generated field $F\subset\mathbb{C}$. Since $\mathcal{O}_E^\times$ is abelian, the restriction of $\widetilde{\rho}$ to $\pi_1(\widehat{S(\mathbb{C})},s)$ factors through the group of coinvariants $\pi_1(\widehat{S(\mathbb{C})},s)_{G_F}$, which is to say that it defines a Galois-invariant element of $H^1_{\mathrm{et}}(S,\mathcal{O}_E^\times)$. As we already discussed, Theorem 1.4 implies that every Galois-invariant element in positive-degree cohomology of a smooth variety is torsion. □

3. p-adic Hodge theory

All the additional structures on cohomology and properties of the monodromy discussed in Section 1 are consequences of the action of a single Frobenius element, that is of the Galois group of a finite field. It is very interesting to consider how these Frobenii elements at different primes interact with each other, and this is far from being fully understood.

A much better understood piece of structure is the action of the Galois group of a local field. Let us fix a prime p, and consider algebraic varieties over a p-adic field K with a perfect residue field. Here by a p-adic field we mean the fraction field $K=\mathrm{Frac}(\mathcal{O}_K)$ of a complete discrete valuation ring $\mathcal{O}_K$ such that residue field $k=\mathcal{O}_K/\mathfrak{m}_K$ has characteristic p, but K has characteristic zero. The main examples of such K are finite extensions of $\mathbb{Q}_p$, but most constructions in p-adic Hodge theory apply to arbitrary K, with the notable exception of the results that rely on finiteness of Galois cohomology and Tate duality.

The subject of p-adic Hodge theory is representations of the Galois group G_K on $\mathbb{Q}_p$-vector spaces, with some of the important examples being provided by étale cohomology of algebraic varieties over K. To appreciate the fact that the action of G_K on $H^i_{\mathrm{et}}(X_{\overline{K}},\mathbb{Q}_\ell)$ contains more information for $\ell=p$ than for other ℓ, consider the following example:

Example 3.1. Let $Y:=\mathbb{P}^1_K/(0\sim\infty)$ be a rational curve with one nodal singularity over a finite extension K of $\mathbb{Q}_p$, as in Example 1.2, and consider the complement $X_a=Y\setminus\{a\}$ to a K-point $a\neq 0,\infty$. We have a short exact sequence of representations of G_K-modules:

$$\mathbb{Z}_\ell\to H^1_{\mathrm{et}}(X_{a,\overline{K}},\mathbb{Z}_\ell)\to\mathbb{Z}_\ell(-1)\tag{3.1}$$

with the first map induced by the embedding $X_a\hookrightarrow Y$, and the second map induced by the pullback along the map $\mathbb{P}^1_K\setminus\{a\}\to X_a$. One can show, by considering the relevant quotient of $\pi_1^{\mathrm{et}}(X_{a,\overline{K}})$ and extending the discussion of Example 1.2, that the extension class of (3.1) in the continuous cohomology $H^1(G_K,\mathbb{Z}_\ell(1))$ of the Galois group is represented by the Kummer cocycle c_a. Here c_a sends an element $g\in G_K$ to the element $c_a(g)\in\mathbb{Z}_\ell$ such that $g(a^{1/\ell^n})=\zeta_{\ell^n}^{c_a(g)}\cdot a^{1/\ell^n}$ for a chosen compatible system of ℓ-power roots of 1 and a in $\overline{K}$.

Varying a, we have the Kummer map

$$a \mapsto c_a : K^\times \to H^1(G_K, \mathbb{Z}_\ell) \tag{3.2}$$

that can be identified with mapping the abelian group $K^\times$ to its ℓ-adic completion $(K^\times)^\wedge_\ell$. But for $\ell \neq p$ a finite index subgroup of $\mathcal{O}_K^\times$ is ℓ-divisible, hence lies in the kernel of the Kummer map. On the other hand, for $\ell = p$ the kernel of Kummer map is finite, which is to say that the isomorphism class of the Galois representation $H^1_{\text{et}}(X_{a,\overline{K}}, \mathbb{Q}_p)$ remembers the value of a, up to being multiplied by a root of unity.

In complex algebraic geometry, an important additional structure on cohomology is the Hodge filtration. From our point of view that all additional structures on cohomology should be recoverable from the arithmetic data of Galois action, it is natural to wonder if the Hodge structure, or at least the Hodge numbers, are recoverable from the Galois action. Over p-adic fields, this is partially achieved by Fontaine's 'mysterious functor':

Theorem 3.2. *There exists a functor*

$$D_{\text{dR}} : \{\textit{continuous } \mathbb{Q}_p\textit{-representations of } G_K\} \to \{\mathbb{Z}\textit{-filtered } K\textit{-vector spaces}\} \tag{3.3}$$

such that for any smooth proper variety X over K for the representation $H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p)$ there is a natural isomorphism $D_{\text{dR}}(H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p)) \simeq H^n_{\text{dR}}(X/K)$ where de Rham cohomology is equipped with the Hodge filtration $F^i := H^n(X, \Omega^{\geq i}_{X/K})$.

In the remainder of this section, we will give a construction of D_{dR} and discuss one application. Even when we are studying a single representation of the Galois group of a p-adic field K, it is useful to be motivated by the Riemann-Hilbert correspondence for local systems on complex manifolds.

Recall that for a connected complex manifold M there is an equivalence

$$\{\text{conjugacy classes of representations } \pi_1(M) \to GL_n(\mathbb{C})\} \simeq \\ \{\text{rank } n \text{ vector bundles with a flat connection on } M\} \tag{3.4}$$

sending a $\mathbb{C}$-local system $\mathbb{L}$ to the vector bundle $\mathbb{L} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_M$. For this construction to make sense, it is crucial that the ring of functions on our space is linear over the ring of coefficients for our local systems. For local systems with coefficients in $\mathbb{C}$ this construction does not immediately apply in algebraic geometry, because a local system $\mathbb{L}$ on $S(\mathbb{C})$ is not locally constant in Zariski or étale topology on the algebraic variety S.

However, for local systems with coefficient in a finite field $\mathbb{F}_p$ a very similar-looking equivalence exists for schemes of characteristic p:

Proposition 3.3 ([Kat73, Proposition 4.1.1])**.** *For a scheme Y over $\mathbb{F}_p$ there is an equivalence*

$$\{\textit{étale local systems of finite-dimensional } \mathbb{F}_p\textit{-vector spaces }\} \simeq \\ \{\textit{vector bundles } E \textit{ on } Y \textit{ with an isomorphism } F_Y^* E \simeq E\} \tag{3.5}$$

The equivalence (3.5) sends an $\mathbb{F}_p$-local system $\mathbb{M}$ to the étale sheaf $\mathbb{M} \otimes_{\mathbb{F}_p} \mathcal{O}_Y$ which gives rise to a vector bundle, because vector bundles satisfy étale descent. Explicitly, if $Y = \operatorname{Spec} A$ is an affine scheme and $\mathbb{M}$ is trivialized by a finite étale G-Galois cover $f : \operatorname{Spec} B \to \operatorname{Spec} A$ then the resulting vector bundle corresponds to the A-module

$$\Gamma(\operatorname{Spec} B, f^*\mathbb{M})^G \tag{3.6}$$

of G-invariants.

Before describing the construction of D_{dR}, let us discuss how far we can get by trying to generalize (3.4) and (3.5) in the most direct way.

Example 3.4. Given a continuous representation $\rho : G_K \to GL(V)$ of the Galois group of a p-adic field K on a finite-dimensional $\mathbb{Q}_p$-vector space V, we can analogously pass to the ind-étale extension $\overline{K}$ of K that tautologically trivializes ρ and consider the space

$$(V \otimes_{\mathbb{Q}_p} \overline{K})^{G_K} \tag{3.7}$$

of invariants with respect to the diagonal action. Using Hilbert's theorem 90 one checks that this space is naturally identified with $V^{\mathrm{fin}} \otimes_{\mathbb{Q}_p} K$ where V^{fin} is the subspace of $v \in V$ such that the Galois orbit $G_K \cdot v$ is finite. However, this space is typically too small, for example for any smooth proper variety X over K that admits a smooth proper model over $\mathcal{O}_K$ the representation $H^n_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)$ has no non-zero finite vectors for any $n > 0$.

Put differently, the faithfully flat extension $K \to \overline{K}$ of course has descent for vector bundles (vector spaces, in this case) but the diagonal action of G_K on $V \otimes_K \overline{K}$ does not provide a descent datum, unless the action of G_K on V factors through a finite quotient. To remedy this, let us choose a G_K-stable $\mathbb{Z}_p$-lattice $V^+ \subset V$, consider finite G_K-modules V^+/p^i for varying i and form invariants $(V^+/p^i \otimes_{\mathbb{Z}/p^i} \mathcal{O}_{\overline{K}}/p^i)^{G_K}$. This $\mathcal{O}_K/p^i$-module is still not guaranteed to be locally free of rank $\dim V$, because the extension $\mathcal{O}_K/p^i \to \mathcal{O}_{\overline{K}}/p^i$ is no longer ind-étale, so a descent datum for it amounts to more than an equivariance for G_K.

Remarkably, this issue essentially disappears if we replace $\mathcal{O}_K$ with a larger extension $\mathcal{O}_{K_\infty}$, where $K_\infty := K(\mu_{p^\infty})$ is the p-cyclotomic extension of K. The following 'almost purity' theorem, proven in this case by Tate and vastly generalized by Faltings and Scholze, is the driving force of most of the results in rational p-adic Hodge theory:

Theorem 3.5 ([Tat67, Proposition 9])**.** *The extension $\mathcal{O}_{K_\infty}/p^i \to \mathcal{O}_{\overline{K}}/p^i$ is almost étale for each i, in particular the p-adic completion of the module of Kähler differentials $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_{K_\infty}}$ is annihilated by every element $a \in \mathcal{O}_{\overline{K}}$ of positive valuation.*

We denote by $\mathbb{C}_p := (\lim\limits_i \mathcal{O}_{\overline{K}}/p^i)[\frac{1}{p}]$ the field obtained by p-adically completing the algebraic closure $\overline{K}$, and similarly let $\widehat{K}_\infty$ be the p-adic completion of the cyclotomic field K_∞. The action of G_K on $\overline{K}$ extends uniquely to a continuous action on $\mathbb{C}_p$.

Corollary 3.6 ([Sen81, Theorem 2])**.** *For any continuous $\mathbb{Q}_p$-representation V of G_K the space*

$$(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{\mathrm{Gal}(\overline{K}/K_\infty)}$$

of invariants has dimension $\dim V$ *over* $\widehat{K}_\infty \simeq \mathbb{C}_p^{\mathrm{Gal}(\overline{K}/K_\infty)}$.

The assignment $V \mapsto (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{\mathrm{Gal}(\overline{K}/K_\infty)}$ comes close to being an analog of (3.4) and (3.5) except that we have not yet used all of the available structure: these invariants still carry residual action of $\Gamma_K := \mathrm{Gal}(K_\infty/K)$ which can be identified with a finite index subgroup of $\mathbb{Z}_p^\times$ via the cyclotomic character map.

A slightly cruder version of the desired functor D_{dR} is obtained by passing to the subspaces on which Γ_K acts through integral powers of the cyclotomic character:

$$D_{\mathrm{HT}}(V) := \bigoplus_{d \in \mathbb{Z}} (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(d))^{G_K} \simeq \mathrm{Hom}_{\Gamma_K}(\bigoplus_{d \in \mathbb{Z}} \chi^d_{\mathrm{cycl}}, (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{\mathrm{Gal}(\overline{K}/K_\infty)}) \tag{3.8}$$

It was conjectured by Tate, and proven by Faltings, that for any smooth proper variety X over K the graded vector space $D_{\mathrm{HT}}(H^n_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p))$ is isomorphic to the n-th Hodge cohomology space $\bigoplus\limits_i H^{n-i}(X, \Omega^i_{X/K})$.

The functor D_{dR} is a refinement of D_{HT} that recovers the de Rham cohomology groups, rather than just the associated graded of the Hodge filtration. It is based on the following refinement of the graded ring $\bigoplus\limits_{d\in\mathbb{Z}} \mathbb{C}_p(d)$.

For a field L of characteristic zero every local Artinian ring A with residue field L has a natural L-algebra structure. This is markedly false in characteristic p, as the example $L = \mathbb{F}_p, A = \mathbb{Z}/p^2$ shows. The field $\mathbb{C}_p$ is of characteristic 0, but it turns out to have interesting non-split nilpotent extensions if one keeps track of the topology on $\mathbb{C}_p$:

Theorem 3.7 ([Fon04, Proposition 3.2], [Col25])**.** *There exists a unique topological $\mathbb{Q}_p$-algebra B^+_{dR} with a pro-nilpotent surjection $\theta : B^+_{\mathrm{dR}} \to \mathbb{C}_p$ with each $B^+_{\mathrm{dR}}/(\ker\theta)^i$ a Banach algebra, such that for any Banach $\mathbb{Q}_p$-algebra A any continuous $\mathbb{Q}_p$-linear surjection $A \twoheadrightarrow \mathbb{C}_p$ with nilpotent kernel factors through a unique continuous map $B^+_{\mathrm{dR}} \to A$. By universality, B^+_{dR} has a natural induced G_K-action, and the i-th graded piece $(\ker\theta)^i/(\ker\theta)^{i+1}$ of the complete filtration defined by the ideal $\ker\theta$ is isomorphic to $\mathbb{C}_p(i)$ as a G_K-module.*

We also denote by B_{dR} the ring (which happens to be a field) $B_{\mathrm{dR}} := B^+_{\mathrm{dR}}[1/t]$ where t is any generator of the principal ideal $\ker\theta$. The functor D_{dR} on any continuous $\mathbb{Q}_p$-representation V of G_K is now defined as

$$D_{\mathrm{dR}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K} \tag{3.9}$$

where invariants are taken with respect to the diagonal action of G_K, and the filtration on $D_{\mathrm{dR}}(V)$ is given by $F^i D_{\mathrm{dR}}(V) := (V \otimes_{\mathbb{Q}_p} t^i \cdot B^+_{\mathrm{dR}})^{G_K}$ This functor, as well as D_{HT}, is not faithful essentially by design:

Example 3.8. For each integer $n \in \mathbb{Z}$ the value $D_{\mathrm{dR}}(\chi^n_{\mathrm{cycl}})$ on the n-th tensor power of the cyclotomic character is a 1-dimensional vector space K, equipped with the filtration given by $F^{-n}K = K, F^{-n+1} = 0$.

For any $a \in \mathbb{Z}_p$ that is sufficiently (depending on the number of roots of unity in K) close to an integer we can also consider the a-th power χ^a_{cycl} of the cyclotomic character. For example, when $p > 2$ or $\sqrt{-1} \in K$, the cyclotomic character admits a square root $\chi^{1/2}_{\mathrm{cycl}}$ and the value $D_{\mathrm{dR}}(\chi^{1/2}_{\mathrm{cycl}})$ is zero.

Indeed, the ring structures on B_{dR} and $\bigoplus\limits_d \mathbb{C}_p(d)$ induce lax monoidal structures on functors D_{dR} and D_{HT}, that is there are natural injective maps $D_{\mathrm{dR}}(V) \otimes D_{\mathrm{dR}}(W) \to D_{\mathrm{dR}}(V \otimes W)$ and similarly for D_{HT}. If $D_{\mathrm{dR}}(\chi^{1/2}_{\mathrm{cycl}})$ was non-zero, the 1-dimensional filtered vector space $D_{\mathrm{dR}}(\chi_{\mathrm{cycl}})$ would admit a square root with respect to the tensor product, which is false because the filtration has the only non-trivial jump at an odd degree.

We have an important subclass of representations on which the functor D_{dR} is faithful:

Definition 3.9. A finite-dimensional $\mathbb{Q}_p$-representation V of G_K is *de Rham* if $\dim_K D_{\mathrm{dR}}(V)$ equals $\dim_{\mathbb{Q}_p} V$.

For each de Rham representation V there is a natural isomorphism $V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \simeq D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}$. In particular, D_{dR} is an exact monoidal functor on the category of de

Rham representations. For any variety X over $\mathbb{Q}_p$ the representations $H^n_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)$ are de Rham for all n, as we will discuss in Theorem 3.11 below.

Deligne's Hodge filtration. Let us finish this introduction to p-adic Hodge theory by demonstrating that functor D_{dR} is versatile enough to recover the Hodge filtration on the cohomology of a non-smooth and non-proper variety from the Galois action on p-adic étale cohomology.

Recall that a mixed $\mathbb{Q}$-Hodge structure is the data of a $\mathbb{Q}$-vector space V together with an increasing *weight* filtration $\ldots \subset W_m V \subset W_{m+1} V \subset \ldots$ by $\mathbb{Q}$-subspaces, and a decreasing filtration F^i by $\mathbb{C}$-subspaces of $V \otimes_{\mathbb{Q}} \mathbb{C}$. These filtrations satisfy an additional compatibility [Del71, Definition 2.3.1]. This additional property has as a remarkable consequence that the category of mixed Hodge structures is abelian and any map $h : (V, W, F) \to (V', W', F')$ of mixed Hodge structures is automatically *strictly* compatible with the filtrations, that is $W'_i V' \cap h(V) = h(W_i V)$ and $h(F^j V_{\mathbb{C}}) = h(V_{\mathbb{C}}) \cap F^j V'_{\mathbb{C}}$.

Deligne constructed a mixed Hodge structure on the cohomology of every complex algebraic variety, generalizing the 'bête' filtration on the de Rham complex in the case of a smooth proper variety:

Corollary 3.10. *For every (possibly non-proper and singular) algebraic variety X over $\mathbb{C}$ there is a natural filtration $F^\bullet$ on $H^n(X(\mathbb{C}), \mathbb{C})$ such that*

(1) For X smooth and proper F^i coincides with the Hodge filtration $H^n(X, \Omega^{\geq i}_X) \subset H^n_{\mathrm{dR}}(X/\mathbb{C}) \simeq H^n(X(\mathbb{C}), \mathbb{C})$ under the comparison isomorphism with de Rham cohomology.

(2) For every map $f : X \to Y$ of algebraic varieties the pullback map $f^* : H^n(Y(\mathbb{C}), \mathbb{C}) \to H^n(X(\mathbb{C}), \mathbb{C})$ is strictly compatible with the Hodge filtration.

It turns out that this Hodge filtration can be also recovered formally from the Galois action on étale cohomology:

Theorem 3.11 ([Kis02, Theorem 3.3])**.** *For any (possibly non-proper and singular) variety X over K there is a natural isomorphism $D_{\mathrm{dR}}(H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p)) \simeq H^n_{\mathrm{dR}}(X/K)$ with Hartshorne's de Rham cohomology such that the natural filtration on D_{dR} corresponds to Deligne's Hodge filtration.*

Remarkably, this formulation makes property (2) of Corollary 3.10 obvious[2]. A map $h : (V, F) \to (V', F')$ of finite-dimensional filtered vector spaces is strictly compatible with the filtrations if and only if h has the same rank as the associated graded map $\mathrm{gr}\, h : \mathrm{gr}^F V \to \mathrm{gr}^{F'} V'$. Any map $f : X \to Y$ of varieties over K induces a morphism $f^*_{\mathrm{et}} : H^n_{\mathrm{et}}(Y_{\overline{K}}, \mathbb{Q}_p) \to H^n_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)$ of de Rham representations, and both functors D_{dR} and D_{HT} are exact on the subcategory of de Rham representations. Kernels of the map $f^*_{\mathrm{dR}} : H^n_{\mathrm{dR}}(Y/K) \to H^n_{\mathrm{dR}}(X/K)$ and its associated graded $\mathrm{gr}\, f^*_{\mathrm{dR}}$ are, respectively, identified with $D_{\mathrm{dR}}(\ker f^*_{\mathrm{et}})$ and $D_{\mathrm{HT}}(\ker f^*_{\mathrm{et}})$ which have the same dimension as $\ker f^*_{\mathrm{et}}$.

[2] According to [Bha, `12:04`], Tate's conjectured Hodge-Tate decomposition was one of the motivations for Deligne to expect to have a Hodge filtration on the cohomology fo any variety that is moreover strictly respected by the pullback along any algebraic map. This led to the discovery of the abelian category of mixed Hodge structures.

4. Local systems of geometric origin over finite fields.

The analog of Conjecture 1 with $\mathbb{C}$ replaced by the algebraic closure $\overline{\mathbb{F}}_p$ of a finite field is settled over curves, as a consequence of the proof of the Langlands correspondence for GL_n over global function fields.

Theorem 4.1. *Let S be a smooth geometrically connected curve over a finite field $\mathbb{F}_q$ of characteristic $p \neq \ell$. A semi-simple continuous representation $\rho : \pi_1^{et}(S_{\overline{\mathbb{F}}_q}) \to GL_n(\overline{\mathbb{Q}}_\ell)$ is of geometric origin if and only if it extends to a representation of $\pi_1^{et}(S_{\mathbb{F}_{q'}})$ for some finite field extension $\mathbb{F}_{q'} \supset \mathbb{F}_q$.*

Proof. Here by 'geometric origin' we mean the condition exactly analogous to Definition 2.4, that is the restriction $\rho|_{\pi_1^{\rm et}(U_{\overline{\mathbb{F}}_q})}$ for some non-empty $U \subset S$ is a subquotient of the monodromy representation on the cohomology of a smooth proper family $f : X \to U_{\overline{\mathbb{F}}_q}$ in some degree.

The 'only if' statement is then a consequence of the fact that any such family can be spread out over $U_{\mathbb{F}_{q'}}$ for some $q' = q^r$. To prove that every representation extendable to the representation over $\mathbb{F}_{q'}$ is of geometric origin, we recall the structure of the proof of the Langlands correspondence for the group GL_n over the field $F = \mathbb{F}_{q'}(S)$ of rational functions on $S_{\mathbb{F}_{q'}}$. For simplicity of notation, we denote $\mathbb{F}_{q'}$ by $\mathbb{F}_q$ in what follows.

The main construction of [Dri77], [Laf02] is an association of an irreducible rank n representation of G_F to every cuspidal automorphic Hecke-eigenform f. The key point for us is that this representation is, by construction, of geometric origin when restricted to $G_{\overline{\mathbb{F}}_q(S)}$. We have a smooth Deligne-Mumford stack $\pi : \mathrm{Sht} \to (S \times S) \setminus \Delta$ over $S \times S \setminus \Delta$ parametrizing rank n shtukas with minimal modifications in opposite directions at two legs, the map down to $(S \times S) \setminus \Delta$ recording the legs.

The morphism π is not proper, so the pushforward $R^n \pi_! \mathbb{Q}_\ell$ is a constructible sheaf on $(S \times S) \setminus \Delta$ that need not be a local system, but it is shown that it admits a direct summand of the form $\mathbb{L}_f \boxtimes \mathbb{L}_f^\vee$ for an irreducible $\overline{\mathbb{Q}}_\ell$-local system $\mathbb{L}_f$ on S that is characterized by the following relation to the automorphic form f. For each closed point $x \in |S|$ the trace $\mathrm{Tr}(\mathrm{Fr}_x : \mathbb{L}_{f,x})$ of the Frobenius automorphism on the stalk of $\mathbb{L}_f$ at x equals the Hecke eigenvalue of f with respect to the Hecke operator at the point x.

Choosing a point $s \in S(\mathbb{F}_q)$ we can restrict $\mathbb{L}_f \boxtimes \mathbb{L}_f^\vee$ to $(S \setminus s) \times \{s\} \subset (S \times S) \setminus \Delta$ to establish $\mathbb{L}_f \otimes (\mathbb{L}_f^\vee)_s$ as a direct summand of the cohomology sheaf of a smooth DM stack over $S \setminus s$.

Using alterations, and perhaps removing more points from $S \setminus s$ one can then show that $\mathbb{L}_f \otimes (\mathbb{L}_f^\vee)_s$ is a direct summand of the cohomology of a smooth proper family $X \to U \subset S \setminus s$. Now base changing to $\overline{\mathbb{F}}_q$, the local system $\mathbb{L}_f \otimes (\mathbb{L}_f^\vee)_s$ becomes isomorphic to $\mathbb{L}_f^{\oplus n}$. Therefore we have shown that $\mathbb{L}_f|_{S_{\overline{\mathbb{F}}_q}}$ is of geometric origin.

One then shows that the resulting mapping $f \mapsto \mathbb{L}_f$ is a bijection between cuspidal automorphic eigenforms and irreducible $\overline{\mathbb{Q}}_\ell$-local systems of rank n on S. In particular, a posteriori every irreducible local system on S arises via this construction. □

Theorem 4.1 is one piece of evidence for Conjecture 1. Given, say, a smooth curve S over $\mathbb{C}$ we can find a relative curve $\mathcal{S}$ over a finitely generated subalgebra $R \subset \mathbb{C}$ such that $\mathcal{S} \times_R \mathbb{C} \simeq S$. An irreducible $\overline{\mathbb{Q}}_\ell$-local system on $S(\mathbb{C})$ which is arithmetic, by definition, extends to $\mathcal{S}_F$ for some finite extension $F \supset \mathrm{Frac}(R)$. Moreover, irreducibility forces (cf. [Pet23, Proposition 6.1]) it to further extend to a local system $\widetilde{\mathbb{L}}$ on $\mathcal{S}$ itself, perhaps after replacing R by an étale extension.

We can now reduce $\mathcal{S}$ to finite field-valued points κ of $\operatorname{Spec} R$ to get curves over various finite fields, and on each of them $\widetilde{\mathbb{L}}|_{\mathcal{S}_\kappa}$ is of geometric origin by Theorem 4.1. However, families of algebraic varieties whose monodromy representations encompass these local systems are completely unrelated across various finite fields.

In some special situations it is possible to choose families of varieties giving rise to the local systems $\widetilde{\mathbb{L}}|_{\mathcal{S}_\kappa}$ to be of uniform geometric type which allows one to produce a family of varieties in characteristic zero. A beautiful example of this strategy working is [ST18, Theorem 1], which proves that rank 2 arithmetic local systems satisfying some additional conditions come from families of elliptic curves. A role in this proof is played by the fact that non-isotrivial families of elliptic curves are completely controlled by their monodromy representation, as we saw in Proposition 1.5.

5. Arithmetic local systems and variations of Hodge structures

Conjecture 1 is an instance of the point of view that all the structures on cohomology of an algebraic variety should be recoverable from the Galois action. Theorem 3.2 allows one to recover the Hodge filtration on de Rham cohomology over a p-adic field, though it stops short of recovering the complex Hodge structure. In this final section, we will see how the situation improves in the presence of a non-trivial monodromy action.

Let S be a smooth algebraic variety over a p-adic field K. We say that an étale $\mathbb{Q}_p$-local system on $S_{\overline{K}}$ is 'p-arithmetic' if it extends to a local system on $S_{K'}$ for some finite extension $K' \supset K$. This is generally a weaker property than being arithmetic, because K' is not a finitely generated field. In Section 3 we discussed a way of recovering de Rham cohomology with its Hodge filtration from the action of G_K on étale cohomology. Works of Faltings, Scholze [Sch13], Liu-Zhu [LZ17] and Diao-Lan-Liu-Zhu [DLLZ23], and many other authors extended this theory to local systems on varieties over K. After the fact, we can produce the following construction for local systems on $S_{\overline{K}}$:

Proposition 5.1. *There is a faithful tensor functor*

$$D : \{\textit{semi-simple } p\textit{-arithmetic } \mathbb{Q}_p\textit{-local systems on } S_{\overline{K}}\} \to \{\textit{vector bundles with a flat connection on } S_{B_{\mathrm{dR}}}\} \tag{5.1}$$

Here $S_{B_{\mathrm{dR}}} = S \times_{\mathbb{Q}_p} B_{\mathrm{dR}}$ is viewed as a smooth variety over the field B_{dR}. Moreover, D becomes fully faithful after extending scalars on the source category from $\mathbb{Q}_p$ to B_{dR}.

Proof. We have [Sch13] a relative version of the functor D_{dR} from Theorem 3.2 which is exact and monoidal on the category of de Rham $\mathbb{Q}_p$-local systems on S:

$$D_{\mathrm{dR}} : \{\text{de Rham } \mathbb{Q}_p\text{-local systems on } S\} \to \left\{ \begin{array}{c} \text{vector bundles } E \text{ with a flat connection } \nabla \text{ on } S \\ \text{and a decreasing filtration } F^i \subset E \text{ satisfying} \\ \nabla(F^i) \subset F^{i-1} \otimes \Omega^1_S \end{array} \right\} \tag{5.2}$$

Moreover, for any de Rham local system $\mathbb{L}$ there is a cohomology comparison isomorphism

$$H^n_{\mathrm{et}}(S_{\overline{K}}, \mathbb{L}) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \simeq H^n_{\mathrm{dR}}(S, D_{\mathrm{dR}}(\mathbb{L})) \otimes_K B_{\mathrm{dR}} \tag{5.3}$$

Let us define a new category $\mathcal{C}_S$ whose objects are de Rham local systems, but we equip them with different spaces of morphisms given by $\operatorname{Hom}_{\mathcal{C}_S}(\mathbb{L}_1, \mathbb{L}_2) =$

$\mathrm{Hom}_{S_{\overline{K}}}(\mathbb{L}_1|_{S_{\overline{K}}}, \mathbb{L}_2|_{S_{\overline{K}}})$. In other words, $\mathcal{C}_S$ is the full subcategory of the category of $\mathbb{Q}_p$-local systems on $S_{\overline{K}}$ consisting of local systems that admit an extension to a de Rham local system on S.

By the above comparison isomorphism (5.3) applied to $n = 0, \mathbb{L} = \mathbb{L}_2 \otimes \mathbb{L}_1^\vee$, functor D_{dR} factors through a functor

$$D : \mathcal{C}_S \to \{\text{flat vector bundles on } S\} \otimes_K B_{\mathrm{dR}} \subset \{\text{flat vector bundles on } S_{B_{\mathrm{dR}}}\} \tag{5.4}$$

where $- \otimes_K B_{\mathrm{dR}}$ denotes passing to the category with the same objects and Hom spaces tensored up to B_{dR} from K.

Since the category of vector bundles with a flat connection on $S_{B_{\mathrm{dR}}}$ is abelian, functor D extends uniquely to an exact functor from the idempotent completion of $\mathcal{C}_S$. Explicitly, this idempotent completion is the full subcategory of the category of local systems on $S_{\overline{K}}$ consisting of local systems that are direct summands of local systems of the form $\mathbb{L}|_{S_{\overline{K}}}$ for some de Rham local system $\mathbb{L}$ on S.

By [Pet23, Theorem 8.1], every semi-simple arithmetic local system on $S_{\overline{K}}$ lies in the idempotent completion of $\mathcal{C}_S$, hence D gives the desired functor. □

If $\mathbb{L}$ is a local system on S of the form $R^n f_* \mathbb{Q}_p$ for a smooth proper morphism $f : X \to S$ then $D(\mathbb{L}|_{S_{\overline{K}}})$ recovers the de Rham cohomology bundle $\mathcal{H}^n_{\mathrm{dR}}(X/S) \otimes_K B_{\mathrm{dR}}$ of the same family. Choosing an embedding $K \subset \mathbb{C}$ we may observe that the flat vector bundle $\mathcal{H}^n_{\mathrm{dR}}(X/S)_{\mathbb{C}}$ on $S_{\mathbb{C}}$ satisfies a rather special property: it underlies a polarizable complex variation of Hodge structures, in the sense of [Sim92, §4].

It would be very interesting to prove that for every arithmetic, or even p-arithmetic, semi-simple local system $\mathbb{L}$ on $S_{\overline{K}}$ the flat vector bundle $D(\mathbb{L}) \otimes_{B_{\mathrm{dR}}} \mathbb{C}$ underlies a polarizable $\mathbb{C}$-VHS for some embedding $B_{\mathrm{dR}} \to \mathbb{C}$. At the moment, we can only show the following:

Proposition 5.2. *For a smooth proper variety S over K and any embedding $K \to \mathbb{C}$ there exists a faithful tensor functor*

$$\{\textit{semi-simple } p\textit{-arithmetic local systems on } S_{\overline{K}}\} \to \{\textit{polarizable } \mathbb{C}\textit{-VHS on } S_{\mathbb{C}}\} \tag{5.5}$$

Proof. Analogously to Proposition 5.1 we have a functor

$$H : \{\text{semi-simple } p\text{-arithmetic local systems on } S_{\overline{K}}\} \to \{\text{Higgs bundles on } S_{\mathbb{C}_p}\} \tag{5.6}$$

constructed out of the functor D_{HT}. In fact, it extends[3] to a functor from all local systems on $S_{\overline{K}}$, given by the p-adic Simpson correspondence in the sense of [Fal05] and [Heu25]. In particular, H lands in curve-semistable Higgs bundles by [HX24, Proposition 9.3.2]. For a de Rham local system $\mathbb{L}$ on S the Higgs bundle $H(\mathbb{L}|_{S_{\overline{K}}})$ is given by the associated graded bundle of the filtration on $D_{\mathrm{dR}}(\mathbb{L})$ with the Higgs field given by the $\mathcal{O}_S$-linear map $\mathrm{gr}^i D_{\mathrm{dR}}(\mathbb{L}) \to \mathrm{gr}^{i-1} D_{\mathrm{dR}}(\mathbb{L}) \otimes \Omega^1_{S/K}$ induced by the Griffiths transverse connection. This implies that vector bundle $H(\mathbb{L}|_{S_{\overline{K}}})$ has vanishing rational Chern classes and there exists an isomorphism

$$(H(\mathbb{L}|_{S_{\overline{K}}}), \theta) \simeq (H(\mathbb{L}|_{S_{\overline{K}}}), \lambda \cdot \theta)$$

of Higgs bundles for every non-zero scalar $\lambda \in K^\times$.

One can check that these properties continue to hold for the values of H on all p-arithmetic local systems. In other words, for each semi-simple p-arithmetic local system $\mathbb{L}$ on $S_{\overline{K}}$ the Higgs bundle $(H(\mathbb{L}), \theta)$ admits a structure of a system of Hodge bundles,

[3] The analogous fact for the functor D from (5.1) is not yet known.

and one can associate to it a polarizable $\mathbb{C}$-VHS on $S_{\mathbb{C}}$ via [Sim92, Corollary 4.2]. Here we chose an extension of the given embedding $K \to \mathbb{C}$ to an embedding $\mathbb{C}_p \to \mathbb{C}$ to form the base change $S_{\mathbb{C}}$. □

The caveat here is that it is completely unclear whether the $\mathbb{C}$-local system output by the functor (5.6) has any relation to the input $\mathbb{Q}_p$-local system, viewed as a local system on the topological space $S(\mathbb{C})$. The existence of such a relation was conjectured in [DLLZ23, Conjecture 1.4], and it would provide further evidence for Conjecture 1.

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